

Layered permutations and rational generating functions

by Bruce Sagan (joint work with Anders Björner)

Partially order the set of all permutations by letting $\pi \leq \sigma$ if and only if σ contains π as a pattern. Consider the subposet $\mathcal{L}$ of layered permutations, i.e., those which can be written as a concatenation

$$\pi = [1, a]^r [a + 1, b]^r [b + 1, c]^r \dots$$

where $[a, b]^r = b, b - 1, b - 2, \dots, a$.

Given $\pi \in \mathcal{L}$ we give an simple expression for the generating function

$$\sum_{\sigma \geq \pi} x^{|\sigma|}$$

where the sum is restricted to those $\sigma \in \mathcal{L}$. Our technique uses the theory of regular languages. (It should be noted that it follows from the work of Albert and Atkinson on simple permutations that this generating function must be rational, and their approach could also be used to calculate it.) As a corollary, we obtain a new Wilf equivalence. Surprisingly, to derive some related generating functions we need identities from the theory of hypergeometric series. This work is closely related to that of Björner and Reutenauer on subword order and we give a common generalization (first considered by Sagan and Vatter).